JOURNAL OF ALGEBRA 86, 422–438 (1984)

On the Existence of a W-Graph for an Irreducible Representation of a Coxeter Group

AKIHIKO GYOJA*

Department of Mathematics, Osaka University, Toyonaka Osaka 560, Japan

Communicated by Jacques Tits

Received July 21, 1981

1.

D. Kazhdan and G. Lusztig defined [5] the notion of W-graph and showed that any irreducible representation of the Coxeter group W (or the corresponding Hecke algebra H_q) of type A_n can be afforded by a W-graph. When the Coxeter group W is of type $B_n = C_n$ $(n \leqslant 5)$, D_4, F_4, H_3, H_4 or $I_2^{(m)}$, a W-graph of any irreducible representation is explicitly given in [1, 3, 7, 9]. So one might expect that any irreducible representation of a finite Coxeter group (or the corresponding Hecke algebra) can be afforded by a W-graph. In this paper, we show that this is the case for any finite Coxeter group (Theorem 2.3) and that there is an irreducible representation of the affine Weyl group of type A_n $(n \geqslant 2)$, which can not be obtained from a W-graph (see Section 3).

We also discuss a characterization of the exceptional representation (Theorem 2.3) and the uniqueness of a W-graph (2.17 and 2.18). In the Appendix, we give a criterion for a triple (X, I, μ) to be a W-graph over C', where X is a set, I (resp. μ) is a mapping $X \to 2^S$ (resp. $X \times X - \text{diagonal} \to C'$) and C' is any commutative ring with an identity.

Notation. For a subring A of B and an A-module M, we put $M^B = M \otimes_A B$. For a set S, we denote its cardinality by $\# S$.

2.

The purpose of this section is to prove Theorem 2.3 below.

Let (W, S) be a Coxeter system. Let C be a commutative ring with an

* This research was supported in part by the Yukawa Foundation and the Grant in Aid for Scientific Research.

0021-8693/84 $3.00

identity and q an element of C. The Hecke algebra $H_q = H_q(W) = H_{q,C}(W)$ is by definition the associative C-algebra with a free C-basis $\{T_w\}_{w \in W}$ and relations

$$T_w T_{w'} = T_{ww'}, \qquad \text{if} \quad l(ww') = l(w) + l(w')$$
$$(T_s + 1)(T_s - q) = 0, \qquad \text{if} \quad s \in S.$$

Here $l(w)$ denotes the length of w.

DEFINITION 2.1 (W-GRAPH [5]). Let C' be a commutative ring with an identity and q an indeterminate over C'. We define a W-graph over C' to be a triple (X, I, μ) of a set X and two mappings $I : X \to 2^S$ and $\mu : X \times X - D \to C'$, where $D = \{(x, x) \mid x \in X\}$. These data are subject to the requirements (2.1.1) and (2.1.2) below. Put $C = C'[q^{1/2}]$. Let E be the free C-module with basis X. Then

$$\text{for any} \quad s \quad \text{in} \quad S \quad \text{and} \quad x \quad \text{in} \quad X$$
$$\bar{T}_s(x) = -x, \qquad \text{if} \quad s \in I(x)$$
$$= qx + q^{1/2} \sum_{\substack{y \in X \\ s \in I(y)}} y\mu(y, x), \qquad \text{if} \quad s \notin I(x) \tag{2.1.1}$$

defines an endomorphism of E and

$$\text{there is a representation } R : H_{q,C}(W) \to \operatorname{End} E \tag{2.1.2}$$

such that $R(T_s) = \bar{T}_s$ for each s in S.

DEFINITION 2.1.3. A W-graph (X, I, μ) is said to be even if there is a mapping $\operatorname{sgn} : X \to \{\pm 1\}$ such that $\mu(y, x) \operatorname{sgn}(x) \operatorname{sgn}(y) = -\mu(y, x)$ for any distinct x, y in X.

In this section, we assume W to be a finite Coxeter group, C' an integral domain of characteristic zero, K' the quotient field of C' and q an indeterminate over K'. We put $K_0 = K'(q)$, $C_0 = C'[q]$, $K = K'(q^{1/2})$ and $C = C'[q^{1/2}]$.

DEFINITION 2.2 (EXCEPTIONAL REPRESENTATION). Let V be an irreducible $H_{q,K}(W)$-module. If there is an $H_{q,K_0}(W)$-module V_0 such that V_0^K is isomorphic to V, then V is said to be non-exceptional. If V is not non-exceptional, then V is said to be exceptional.

Remark 2.2.1. Let W be a finite irreducible Coxeter group. The exceptional representations are:

(1) two 512 dimensional irreducible representations (W is of type E_7),

(2) four 4096 dimensional irreducible representations (W is of type E_8),

(3) two 4 dimensional irreducible representations (W is of type H_3),

(4) four 16 dimensional irreducible representations which are not self-dual (W is of type H_4).

If $W = W_1 \times \cdots \times W_m$ and $R = R_1 \otimes \cdots \otimes R_m$, where R_i are irreducible representations of $H_{q,K}(W_i)$, then R is exceptional iff at least one of the R_i's is exceptional. If W is crystallographic, our definition of exceptional representation essentially coincides with that of Lusztig [6]. (See [2].)

2.3. Let W be a finite Coxeter group, K' its splitting field, C' the ring of integers in K', $K = K'(q^{1/2})$ and $C = C'[q^{1/2}]$.

THEOREM. (1) *Every irreducible $H_{q,K}(W)$-module is afforded by a W-graph over C'.*

(2) *An irreducible $H_{q,K}(W)$-module is afforded by an even W-graph over C', iff it is not exceptional.*

The rest of this section is devoted to the proof of this theorem. Since the W-graphs of non-crystallographic Coxeter groups are explicitly known, we may assume W to be crystallographic, $K' = \mathbf{Q}$ and $C' = \mathbf{Z}$. We consider only W-graphs over $\mathbf{Z}$, so, in the following in this section "a W-graph" means "a W-graph over $\mathbf{Z}$."

DEFINITION 2.4. Let $H = H(W)$ be an algebra over $\mathbf{Z}$ defined by the following presentation: it has generators $s(0)$ and $s(1)$ for each s in S; these generators satisfy the relations

$$\begin{aligned} s(0)^2 &= s(0) \\ s(0)\,s'(0) &= s'(0)\,s(0) \\ s(0)\,s(1) &= s(1) \\ s(1)\,s(0) &= 0 \end{aligned} \tag{2.4.1}$$

and

if we define an element T_s of $H(W) \otimes \mathbf{Z}[q^{1/2}]$ by

$$T_s = -s(0) + q(1 - s(0)) + q^{1/2} s(1) \tag{2.4.2}$$

then

$$\underbrace{T_s T_{s'} T_s \cdots}_{m \text{ factors}} = \underbrace{T_{s'} T_s T_{s'} \cdots}_{m \text{ factors}}.$$

Here s and s' are elements of S and m is the order of ss'.

Remark 2.4.3. Since the element T_s of $H(W) \otimes \mathbf{Z}[q^{1/2}]$ satisfies the identity $(T_s + 1)(T_s - q) = 0$, we get an algebra homomorphism $f_q : H_{q,\mathbf{Z}[q^{1/2}]}(W) \to H(W) \otimes \mathbf{Z}[q^{1/2}]$.

Remark 2.4.4. Let ω be a linear character of $H(W)$. Then ω induces a linear character of $H_q(W)$. Hence

$$\omega(s(1)) = 0$$
$$\omega(s(0)) = 0, 1$$

and

$$\omega(s(0)) = \omega(s'(0)) \qquad \text{if} \quad s \underset{W}{\sim} s'.$$

Conversely, if these conditions are satisfied, ω defines a linear character. Thus there exists a bijection

$$\mathrm{Hom}(H_{q,C}(W), C) \leftarrow \mathrm{Hom}(H(W), \mathbf{Z}).$$

2.5. Define an element A_I of $H(W)$ for each subset I of S by

$$A_I = \Big(\prod_{s \in I} s(0)\Big) \Big(\prod_{s \notin I} 1 - s(0)\Big).$$

LEMMA. *Let J and K be two subsets of S such that $J \not\subset K$ and s an element of $J - K$. Then the element $A_J s(1) A_K$ does not depend on the choice of s. We denote this element of $H(W)$ by $B_{J \not\subset K}$.*

Proof. Let s and s' be two elements in S such that $\mathrm{ord}(ss') = m$. Then

$$\begin{aligned} s(0)\, s'(0) &\underbrace{T_s T_{s'} T_s \cdots}_{m \text{ factors}} \\ &= s'(0)(-s(0) + q^{1/2} s(1))\, T_{s'} T_s \cdots \\ &= s'(0)\, q^{1/2} s(1)\, q^{m-1}(1 - s'(0)) + o(q^{m-(1/2)}). \end{aligned}$$

Hence by (2.4.1) and (2.4.2), we get

$$s'(0)\, s(1)(1 - s'(0)) = s(0)\, s'(1)(1 - s(0)).$$

Now take s and s' from $J-K$. Then $A_J s(0) = A_J s'(0) = A_J$ and $(1-s(0))A_K = (1-s'(0))A_K = A_K$. Hence $A_J s(1) A_K = A_J s'(1) A_K$.

DEFINITION 2.6. We define an automorphism σ of $H = H(W)$ by

$$\sigma s(i) = (-1)^i s(i) \qquad (s \in S, i = 0, 1).$$

Let σ' be the generator of the cyclic group of order 2. Put $\tilde{H}(W) = H(W) \otimes \mathbf{Z}\langle\sigma'\rangle$. We define the multiplication by

$$(h_1 \otimes \sigma'^n)(h_2 \otimes \sigma'^m) = h_1 \sigma^n(h_2) \otimes \sigma'^{n+m}$$

for h_1, h_2 in $H(W)$. In the following we write σ for σ' and $h\sigma$ for $h \otimes \sigma$.

Remark 2.7. Let V be an $H(W)$-module. (Here and below, we consider only an $H(W)$-module which is free and of finite rank as a $\mathbf{Z}$-module.) Put $V_I = A_I V$. Then $V = \bigoplus_{I \subset S} V_I$ and A_I is the projection onto V_I. Let $\{v_{I,i}\}_i$ be a free basis of V_I, $B_{J \not\subset K} v_{K,k} = \sum_j v_{J,j} \mu(v_{J,j}, v_{K,k})$, $I(v_{I,i}) = I$ and $X = \{v_{I,i}\}_{I,i}$. Then $\mu(v_{J,j}, v_{K,k})$ are integers and (X, I, μ) is a W-graph. This W-graph gives an $H_{q,C}(W)$-module which is isomorphic to $V^C|_{H_{q,C}(W)}$.

Conversely a W-graph gives an $H(W)$-module. In fact, let (X, I, μ) be a W-graph. Let V be a free $\mathbf{Z}$-module with a basis X. We define a left $H(W)$-module structure on V as follows:

$$s(0)\,x = \begin{cases} x, & \text{if } s \in I(x) \\ 0, & \text{if } s \notin I(x) \end{cases}$$

$$s(1)\,x = \begin{cases} \displaystyle\sum_{\substack{y \in X \\ s \in I(y)}} y\mu(y, x), & \text{if } s \notin I(x) \\ 0, & \text{if } s \in I(x). \end{cases}$$

Moreover if the W-graph (X, I, μ) is even, then we can define a left $\tilde{H}(W)$-module structure on V by

$$\sigma x = \operatorname{sgn}(x)\, x.$$

(See Definition 2.1.3 for the definition of sgn.)

LEMMA 2.8. *Let V be an $\tilde{H}(W)$-module. Then there is an even W-graph which gives an $H(W)^{\mathbf{Q}}$-module isomorphic to $V^{\mathbf{Q}}|_{H(W)^{\mathbf{Q}}}$.*

Proof. Let $V^{\mathbf{Q}\pm}$ be the eigenspace of σ which belongs to the eigenvalue ± 1. Put $V^{\pm} = V^{\mathbf{Q}\pm} \cap V$ and $V_1 = V^+ \oplus V^-$. Then V_1 is an $H(W)$-module and $V_1^{\mathbf{Q}} = V^{\mathbf{Q}}$. Let $X_I^{\pm}$ be a free basis for $A_I V^{\pm}$ and $X = \bigcup_I (X_I^+ \cup X_I^-)$. Then in the same way as in Remark 2.7, we can construct a W-graph (X, I, μ). Define a mapping $\mathrm{sgn} : X \to \{\pm 1\}$ by $\mathrm{sgn}|_{X_I^{\pm}} = \pm 1$. Since $B_{J \not\subset K}(A_K V^{\pm}) \subset A_J V^{\mp}$, we get the required relation

$$\mu(y, x)\,\mathrm{sgn}(y)\,\mathrm{sgn}(x) = -\mu(y, x) \qquad (x, y \in X).$$

DEFINITION 2.9. Let E be the free $\mathbf{Z}$-module with a basis $(e_w)_{w \in W}$. We define a left (resp. right) $\tilde{H}(W)$-module structure on E as follows:

$$s(0)\, e_w = \begin{cases} e_w, & \text{if } s \in \mathscr{L}(w) \\ 0, & \text{if } s \notin \mathscr{L}(w) \end{cases}$$

$$s(1)\, e_w = \begin{cases} \sum\limits_{\substack{y \sim_{LR} w \\ s \in \mathscr{L}(y)}} e_y \mu(y, w), & \text{if } s \notin \mathscr{L}(w) \\ 0, & \text{if } s \in \mathscr{L}(w) \end{cases}$$

$$\sigma e_w = \mathrm{sgn}(w)\, e_w$$

(resp.

$$e_w s(0) = \begin{cases} e_w, & \text{if } s \in \mathscr{R}(w) \\ 0, & \text{if } s \notin \mathscr{R}(w) \end{cases}$$

$$e_w s(1) = \begin{cases} \sum\limits_{\substack{y \sim_{LR} w \\ s \in \mathscr{R}(y)}} \mu(w, y)\, e_y, & \text{if } s \notin \mathscr{R}(w) \\ 0, & \text{if } s \in \mathscr{R}(w) \end{cases}$$

$$e_w \sigma = \mathrm{sgn}(w)\, e_w).$$

(See [5] for the definition of $\mu(y, w)$, $\mathscr{L}(w)$, $\mathscr{R}(w)$ and $\sim_{LR}$.)

This left $\tilde{H}(W)$-module structure on E defines an algebra homomorphism $\tilde{H}(W) \to \mathrm{End}(E)$. Denote the image of $H(W)$ (resp. $\tilde{H}(W)$) by $H'(W)$ (resp. $\tilde{H}'(W)$). Let $c : \tilde{H}(W) \to \tilde{H}'(W)$ be the natural homomorphism.

Remark 2.10. Since the element T_s of $H(W) \otimes \mathbf{Z}[q^{1/2}]$ acts on E^K by

$$T_s e_w = \begin{cases} -e_w, & \text{if } s \in \mathscr{L}(w) \\ q e_w + q^{1/2} \sum\limits_{\substack{y \sim_{LR} w \\ s \in \mathscr{L}(y)}} e_y \mu(y, w), & \text{if } s \notin \mathscr{L}(w) \end{cases} \qquad (2.10.1)$$

and

$$e_w T_s = \begin{cases} -e_w, & \text{if } \ s \in \mathscr{R}(w) \\ qe_w + q^{1/2} \displaystyle\sum_{\substack{y \sim_{LR} w \\ s \in \mathscr{R}(y)}} \mu(w, y)\, e_y, & \text{if } \ s \notin \mathscr{R}(w), \end{cases} \tag{2.10.2}$$

the left and right $H(W)$-module structure on E induces a left and right $H_{q,K}(W)$-module structure on E^K, which is known to be isomorphic to the two-sided regular representation [7]. Hence f_q is injective. (See (2.4.3) for f_q.) In the following we regard the Hecke algebra $H_{q,C}(W)$ as a subalgebra of $H(W)^C$.

LEMMA 2.11. *The left and right $H(W)$-module structures on E commute.*

Proof. Let q_1 and q_2 be two square integers. Then E has a left $H_{q_1,\mathbf{Z}}(W)$-module structure (resp. a right $H_{q_2,\mathbf{Z}}(W)$-module structure) defined by (2.10.1) (resp. (2.10.2)). Then in the same way as in the proof of [7, Lemma 2.3], we can prove that the left $H_{q_1,\mathbf{Z}}(W)$-module structure and the right $H_{q_2,\mathbf{Z}}(W)$-module structure on E commute. The algebra generated by the endomorphisms of E defined by the left (resp. right) action of $H(W)$ is generated by the endomorphisms defined by the left (resp. right) action of $\bigcup_{r=\text{square}} H_{r,\mathbf{Z}}(W)$. Hence the left and right $H(W)$-module structures on E commute.

Remark 2.12. Let $\mathrm{End}_0(E^K)$ be the algebra of endomorphisms of the right $H_{q,K}(W)$-module E^K. Since the two-sided $H_{q,K}(W)$-module E^K is isomorphic to the two-sided regular representation of $H_{q,K}(W)$ [7], the left $H_{q,K}(W)$-module structure on E^K gives rise to an isomorphism $a = a_q$: $H_{q,K}(W) \simeq \mathrm{End}_0(E^K)$ of algebras. The left $\tilde{H}(W)$-module structure on E gives rise to an algebra homomorphism $b : \tilde{H}(W) \to \mathrm{End}(E)$ and

$$a^{-1} b|_{H_{q,K}(W)} = \text{identity}. \tag{2.12.1}$$

This homomorphism b induces a homomorphism $b' : H'(W) \to \mathrm{End}(E)$. Thus we get the commutative diagram

$$\begin{tikzcd}
H'(W)^K \arrow[drr, "b'^K"] & & \\
H(W)^K \arrow[u, "c^K"] \arrow[rr, "b^K"] & & \mathrm{End}_0(E^K) \\
H_{q,K}(W) \arrow[u, hook, "f_q"] \arrow[rr, "\text{identity}"] & & H_{q,K}(W) \arrow[u, "a", "\wr"']
\end{tikzcd}$$

(see f_q for (2.4.3) and c for (2.9)). To simplify notation we write b (resp. b', c) for b^K (resp. b'^K, c^K).

2.3$'$. By 2.7 and 2.8, to give an $H(W)$-module (resp. $\tilde{H}(W)$-module) is to give a W-graph (resp. an even W-graph) and vice versa. Hence to prove Theorem 2.3 it suffices to show the following theorem.

THEOREM. *Let W be a finite crystallographic Coxeter group.*

(1) *Every $H'(W)^K$-module (resp. $\tilde{H}'(W)^K$-module) can be obtained as U^K for some $H'(W)$-module (resp. $H'(W)$-module) U.*

(2) *Every irreducible $H_{q,K}(W)$-module can be obtained as $c^*V^K|_{H_{q,K}(W)}$ for some $H'(W)$-module V.*

(3) *Let V be an $H(W)$-module such that $V^K|_{H_{q,K}(W)}$ is irreducible and exceptional. Then V cannot be extended to an $\tilde{H}(W)$-module.*

(4) *Every irreducible non-exceptional $H_{q,K}(W)$-module can be obtained as $c^*V^K|_{H_{q,K}(W)}$ for some $\tilde{H}'(W)$-module V.*

2.13. *Proof of* 2.3$'$(1). Let V be an $H'(W)^K$-module. Since K is a rational function field over $\mathbf{Q}$, there is an $H'(W)^{\mathbf{Q}}$-module V_1 such that V_1^K is isomorphic to V. Let V_2 be a lattice of V_1. Then $H'(W)\,V_2 = U$ is also a lattice of V_1. This $H'(W)$-module U has the desired property. The statement about $\tilde{H}'(W)$ can be proved in the same way.

Remark 2.13.1. In the preliminary draft of this paper, the author was indifferent to the integrality of μ of a W-graph (X, I, μ) and proved that every irreducible $H_q(W)$-module is afforded by a W-graph over $\mathbf{C}$. It was G. Lusztig who first proved this integrality (letter to the author, 20 October 1981). His proof is different from the proof given here. The author would like to thank G. Lusztig for this comment.

2.14. *Proof of* 2.3$'$(2). Let V be an irreducible $H_{q,K}(W)$-module. By (1), it suffices to prove that there is an $H'(W)^K$-module V_1 such that $c^*V_1|_{H_{q,K}(W)}$ is isomorphic to V. We define an $H'(W)^K$-module structure on V via the homomorphism $a^{-1}b' : H'(W)^K \to H_{q,K}(W)$. Since $a^{-1}b|_{H_{q,K}(W)}$ is an identity, this $H'(W)^K$-module has the desired property.

2.15. *Proof of* 2.3$'$(3). Let V be an $H(W)$-module such that $V^K|_{H_{q,K}(W)}$ is irreducible and exceptional. Assume that V can be extended to an $\tilde{H}(W)$-module. Then by 2.8, there is an even W-graph (X, I, μ) which gives an $H(W)^{\mathbf{Q}}$-module isomorphic to $V^{\mathbf{Q}}$. Put

$$V' = \sum_{\substack{x \in X \\ \mathrm{sgn}(x) = 1}} K_0 x + \sum_{\substack{x \in X \\ \mathrm{sgn}(x) = -1}} K_0 q^{1/2} x.$$

Then V' is an $H_{q,K_0}(W)$-module such that V'^K is isomorphic to V. This contradicts to the assumption that $V^K|_{H_{q,K}(W)}$ is exceptional.

2.16. *Proof of* 2.3′(4). Let j be the field automorphism of $K = \mathbf{Q}(q^{1/2})$ defined by $j(q^{1/2}) = -q^{1/2}$. Put

$$E' = \sum_{w \in W} K_0 q^{l(w)/2} e_w.$$

Then E' is an $H_{q,K_0}(W)$-module such that $E'^K = E$. For an element $e' \otimes k$ of E'^K, put $j(e' \otimes k) = e' \otimes j(k)$. Let V be an irreducible non-exceptional $H_{q,K}(W)$-module. We may assume V to be an $H_{q,K}(W)$-submodule of E. Then E' has an $H_{q,K_0}(W)$-submodule V' such that V'^K is isomorphic to V. Since the subspace V'^K of E'^K is j-stable, V is $\langle H_{q,K}(W), j\rangle$-module. Then an $\langle H'(W)^K, j\rangle$-module structure on V can be defined via the homomorphism $a^{-1}b' : H'(W)^K \to H_{q,K}(W)$. By (1), it suffices to prove that $\sigma \mapsto j$ defines an $\tilde{H}'(W)^K$-module structure on V. Since $j^2 = 1$, it suffices to prove

$$j \cdot a^{-1}b(s(i)) \cdot j = a^{-1}b((-1)^i s(i)).$$

Since these two elements induce K-linear endomorphisms of E^K, it suffices to prove

$$js(i)j(q^{l(w)/2}e_w) = (-1)^i s(i)(q^{l(w)/2}e_w).$$

If $i = 1$ and $s \notin \mathscr{L}(w)$, the left-hand side is equal to

$$\begin{aligned} &js(1)(q^{l(w)/2}e_w) \\ &\quad = j \sum_{\substack{y \sim_{LR} w \\ s \in \mathscr{L}(y)}} q^{(l(w)-l(y))/2}(q^{l(y)/2}e_y)\,\mu(y, w) \\ &\quad = - \sum_{\substack{y \sim_{LR} w \\ s \in \mathscr{L}(y)}} q^{(l(w)-l(y))/2}(q^{l(y)/2}e_y)\,\mu(y, w) \\ &\quad = -s(1)(q^{l(w)/2}e_w), \end{aligned}$$

since $l(w) - l(y)$ is odd if $\mu(y, w) \neq 0$. The other cases are less difficult.

2.17. In this paragraph, we discuss a kind of uniqueness of a W-graph. Let D be a two-sided cell, R a right cell, $E_D = \bigoplus_{w \in D} \mathbf{Z}e_w$ and $E_R = \bigoplus_{w \in R} \mathbf{Z}e_w$.

THEOREM. *Let V be an irreducible $H_{q,K}(W)$-module appearing in E_D^K. Then V has a cannonical decomposition into subspaces*

$$V = \bigoplus_{\substack{R \subset D \\ R \text{ right cell}}} V_R \tag{2.17.1}$$

such that the dimension of V_R is equal to the multiplicity of V in the right $H_{q,K}(W)$-module E_R^K and that

$$A_I V = \bigoplus_{\substack{R \subset D \\ \mathscr{L}(R) = I}} V_R. \tag{2.17.2}$$

Here $\mathscr{L}(R) = \mathscr{L}(w)$ $(w \in R)$, which is well defined (see [5]).

Proof. Let $\hat{V}$ be an irreducible right $H_{q,K}(W)$-module such that $\mathrm{Hom}(\hat{V}, E_D^K)$ is isomorphic to V. Then the subspaces $\mathrm{Hom}(\hat{V}, E_R^K)$ have the desired property.

Remark 2.17.3. Let X_R be a basis of V_R, $X_I = \bigcup_{\mathscr{L}(R)=I} X_R$ and $X = \bigcup X_I$. Then as in (2.7), we can construct a W-graph. If every E_R^K is multiplicity free, then such a W-graph is unique up to diagonal equivalence [4, Definition 2.2]. In fact, if W is of type A_n, H_3 or $I_2^{(m)}$, then E_R^K is multiplicity free. Assume that Lusztig's conjecture that the right cell representations coincide with the cell representations [8] is valid. Then E_R^K is multiplicity free if W is of type $B_n = C_n$ or D_n.

Remark 2.18. Since $b : H(W)^K \to \mathrm{End}_0(E^K)$ is surjective by (2.12.1), the kernel of b contains the Jacobson radical of $H(W)^K$.

Problem. Does the kernel of b coincide with the Jacobson radical of $H(W)^K$?

This problem is equivalent to the following problem. Assume that V_1 and V_2 are two completely reducible $H(W)^K$-modules such that $V_1|_{H_{q,K}(W)} \simeq V_2|_{H_{q,K}(W)}$.

Problem. Are such $H(W)^K$-modules V_1 and V_2 isomorphic to each other?

For such $H(W)^K$-modules, we can prove that

$$\dim A_I V_1 = \dim A_I V_2 \qquad (I \subset S).$$

In fact, for an $H(W)^K$-module V, put $d_I(V) = \dim\{v \in V \mid T_s v = -v \ (s \in I)\}$. Then

$$d_I(V) = \sum_{J \supset I} \dim A_J V$$

and

$$\dim A_I V = \sum_{J \supset I} (-1)^{\#(J-I)} d_J(V).$$

3.

Let W_a be the affine Weyl group of type A_n $(n \geqslant 2)$ and $S_a = \{s_i\}_{i \in \mathbf{Z}/(n+1)}$ the set of canonical generators such that $(s_i s_{i+1})^3 = 1$. Let $T_i = T_{s_i}$, $S = \{s_1, ..., s_n\}$ and W be the group generated by S. Let $A : H_q(W_a) \to H_q(W)$ be the algebra homomorphism defined by

$$\begin{aligned} AT_0 &= T_1 T_2 \cdots T_{n-1} T_n T_{n-1}^{-1} \cdots T_2^{-1} T_1^{-1} \\ AT_i &= T_i \qquad (1 \leqslant i \leqslant n). \end{aligned}$$

(See [4, Theorem 3.1].) Let R be the irreducible representation of $H_q(W)$ given by the W-graph (X, I, μ) such that $X = (x_i)_{1 \leqslant i \leqslant n}$, $I(x_i) = \{s_i\}$ and

$$\mu(x_i, x_j) = \begin{cases} 1, & \text{if } \ i - j = \pm 1 \\ 0, & \text{if } \ \text{otherwise.} \end{cases}$$

With respect to the basis $\{q^{i/2} x_i\}_{1 \leqslant i \leqslant n}$, the operators T_i $(0 \leqslant i \leqslant n)$ are represented by the following matrices:

$$\bar{T}_0 = \begin{pmatrix} 0 & & & & -1 \\ -q & q & & & -1 \\ \vdots & & \ddots & & \vdots \\ -q & & & q & -1 \\ -q & & & & q-1 \end{pmatrix},$$

$$T_i = \begin{pmatrix} q & & & & & & \\ & \ddots & & & & & \\ & & q & & & & \\ & & 1 & -1 & q & & \\ & & & & q & & \\ & & & & & \ddots & \\ & & & & & & q \end{pmatrix} \qquad (1 \leqslant i \leqslant n),$$

where -1 appears in the ith row of $\bar{T}_i$ $(1 \leqslant i \leqslant n)$. Put $U_i = \bar{T}_i|_{q=0}$ $(0 \leqslant i \leqslant n)$. Then $U_i^2 = -U_i$ for any i and $U_i U_j = 0$ unless $i - j = 1$ or 0. Let $X = \coprod X_i$ be a proper subset of S, where $X_i = \{s_{a_i}, s_{a_i+1}, ..., s_{b_i}\}$ and for any $s \in X_i$, $s' \in X_{i'}$ $(i \neq i')$, $ss' = s's$. For a finite-dimensional representation R' of $H_q(W_a)$ and a subset W' of W_a, put

$$L(t, W', R') = \sum_{w \in W'} R'(T_w)\, t^{l(w)}$$

and $L(t, R') = L(t, W_a, R')$. (See [4].) Every element w of W_{X_i} can be expressed uniquely as

$$w = (s_{b_i} s_{b_i+1} \cdots s_{b_i+k_1})(s_{b_i-1} s_{(b_i-1)+1} \cdots s_{(b_i-1)+k_2}) \\ \cdots (s_{a_i} s_{a_i+1} \cdots s_{a_i+k_m})$$

with a sequence $\{k_i\}_{i=1}^{m}$ of integers such that $0 \leqslant k_i < i$. Here W_X denotes the subgroup of W_a which is generated by X and $m = b_i - a_i + 1$. Moreover, this product always gives a reduced expression. (See [4, Theorem 3.3].) Hence

$$\begin{aligned} L(t, W_X, R \circ A)|_{q=0} &= \prod_i L(t, W_{X_i}, R \circ A)|_{q=0} \\ &= \prod_i (1 + tU_{b_i})(1 + tU_{b_i-1} + t^2 U_{b_i-1} U_{b_i}) \\ &\qquad \cdots (1 + tU_{a_i} + \cdots + t^{b_i - a_i + 1} U_{a_i} \cdots U_{b_i}) \\ &= \prod_i (1 + tU_{b_i})(1 + tU_{b_i-1}) \cdots (1 + tU_{a_i}). \end{aligned}$$

Since $(1 + tU_i)^{-1} = 1 - t(1-t)^{-1} U_i$,

$$\begin{aligned} &L(t, R \circ A)^{-1}|_{q=0} \\ &\quad = -\sum_{X \subsetneq S_a} (-1)^{|X|} L(t, W_X, R \circ A)^{-1}|_{q=0} \\ &\quad = -\sum_{X \subsetneq S_a} (-1)^{|X|} \prod_i ((1 + tU_{a_i})^{-1} \cdots (1 + tU_{b_i})^{-1}) \\ &\quad = -\sum_{X \subsetneq S_a} (-1)^{|X|} \prod_i (1 - t(1-t)^{-1} U_{a_i}) \cdots (1 - t(1-t)^{-1} U_{b_i}) \\ &\quad = -\sum_{X \subsetneq S_a} (-1)^{|X|} \left(1 - t(1-t)^{-1} \sum_{s_i \in X} U_i\right) \\ &\quad = (-1)^{|S_a|} \left(1 - t(1-t)^{-1} \sum_{i=0}^{n} U_i\right). \end{aligned}$$

Hence

$$\det L(t, R \circ A)|_{q=0} = (1-t)^n / (1 + t + \cdots t^n).$$

Assume that the irreducible representation $R \circ A$ can be obtained from a W-graph (X, I, μ). Since $R \circ A$ is irreducible, $I(x) \neq S_a$ $(x \in X)$. Hence

$$\det L(t, R \circ A)|_{q=0} = \prod_{x \in X} L(t, W_{I(x)}, \mathrm{sgn})$$

is a polynomial in t. (See [4, proof of Theorem 2.4(2)].) This is absurd. Hence $R \circ A$ can not be obtained from a W-graph.

Appendix

In the Appendix, we give another presentation of $H(W)$, which does not invoke the indeterminate q. This presentation can be considered as a criterion for a triple (X, I, μ) to be a W-graph over C', where X is a set, I (resp. μ) is a mapping $X \to 2^S$ (resp. $X \times X -$ diagonal $\to C'$) and C' is any commutative ring with an identity. To simplify notations, we consider the opposed algebra $H(W)^\circ$ of $H(W)$.

For two elements s, t of S such that $\mathrm{ord}(st) = m$, we put

$$\bar{A}_X = \Bigg(\sum_{\substack{I \subset S \\ I \cap \{s,t\} = X}} A_I \Bigg)^\circ$$

$$\bar{B}_{Y \not\supset X} = \Bigg(\sum_{\substack{J,K \subset S \\ J \cap \{s,t\} = X \\ K \cap \{s,t\} = Y}} B_{J \not\subset K} \Bigg)^\circ,$$

where X, Y are subsets of $\{s, t\}$. Put

$$a(i) = \begin{cases} s, & \text{if } i \text{ is odd} \\ t, & \text{if } i \text{ is even} \end{cases}$$

$$b(i) = a(i+1)$$

$$(s, 0) = \bar{A}_{\{s\}}, \qquad (t, 0) = \bar{A}_{\{t\}}$$

$$(s, 1) = \bar{B}_{\{t\} \not\supset \{s\}}, \qquad (t, 1) = \bar{B}_{\{s\} \not\supset \{t\}}$$

$$(s, 2) = \bar{A}_{\{t\}}, \qquad (t, 2) = \bar{A}_{\{s\}}$$

$$(s, 1') = \bar{B}_{\phi \not\supset \{s\}}, \qquad (t, 1') = \bar{B}_{\phi \not\supset \{t\}}$$

$$(s, 1'') = \bar{B}_{\{t\} \not\supset \{s,t\}}, \qquad (t, 1'') = \bar{B}_{\{s\} \not\supset \{s,t\}}.$$

For a mapping $f\colon \{1, 2, \ldots, k\} \to \{0, 1, 1', 1'', 2\}$, put

$$(a,f) = (a(1), f(1))(a(2), f(2)) \cdots (a(k), f(k))$$

$$(b,f) = (b(1), f(1))(b(2), f(2)) \cdots (b(k), f(k)).$$

Define mappings f_k and 1_k by

$$f_k(i) = \begin{cases} 1' & (i=1) \\ 1 & (1<i<k) \\ 1'' & (i=k) \end{cases}$$

and

$$1_k(i) = 1 \qquad (1 \leqslant i \leqslant k).$$

PROPOSITION A1. *The following equalities hold:*

(A2) $$\bar{A}_X \bar{A}_Y = \begin{cases} \bar{A}_X, & \text{if } X = Y \\ 0, & \text{if } X \neq Y \end{cases}$$

$$\bar{A}_X \bar{B}_{Y \not\supset Z} = \begin{cases} \bar{B}_{Y \not\supset Z}, & \text{if } X = Y \\ 0, & \text{if } X \neq Y \end{cases}$$

$$\bar{B}_{Y \not\supset Z} \bar{A}_X = \begin{cases} \bar{B}_{Y \not\supset Z}, & \text{if } X = Z \\ 0, & \text{if } X \neq Z \end{cases}$$

$(A3)_k$ $$(a, f_k) = (b, f_k) \qquad (2 \leqslant k \leqslant m)$$

$(A4)_a$ $$\sum_{k=0}^{[(m-1)/2]} (-1)^k \binom{m-1-k}{k} (a, 1_{m-1-2k}) = 0$$

$(A4)_b$ $$\sum_{k=0}^{[(m-1)/2]} (-1)^k \binom{m-1-k}{k} (b, 1_{m-1-2k}) = 0.$$

Conversely, these are the defining relation of $H(W)$.

Proof. Let $F(m)$ denote the set of mappings $f\colon \{1, 2, ..., m\} \to \{0, 1, 2\}$ such that for any $i = 1, 2, ..., m-1$

$$f(i+1) = 0, \qquad \text{iff} \quad f(i) = 2.$$

Put $|f| = \sum_{j=1}^{m} f(j)$ and $m(f, i) = \#\{1 \leqslant j \leqslant m \mid f(j) = i\}$. It is easy to see that the relations (2.4.1) and (2.4.2) are equivalent to (A2) and

$(A5)_{X,Y}$ $$\bar{A}_X T^\circ_{a(1)} T^\circ_{a(2)} \cdots T^\circ_{a(m)} \bar{A}_Y = \bar{A}_X T^\circ_{b(1)} T^\circ_{b(2)} \cdots T^\circ_{b(m)} \bar{A}_Y \qquad (X, Y \subset \{s, t\}).$$

Hence it suffices to prove that the relations (A5) are equivalent to (A3) and (A4). We shall prove only that "$(A5)_{\{s\},\{t\}}$ and $(A5)_{\{s\},\{s\}}$" is equivalent to $(A4)_b$. The remaining are not more difficult. We have

$$\bar{A}_{\{s\}} T^\circ_{a(1)} T^\circ_{a(2)} \cdots T^\circ_{a(m)} (\bar{A}_{\{s\}} + \bar{A}_{\{t\}})$$
$$= \sum_{\substack{f \in F(m) \\ f(1)=0}} (-1)^{m(f,0)} q^{|f|/2} (a,f)$$
$$= q^{m/2} \sum_{\substack{f \in F(m) \\ f(1)=0, f(m)=2}} (-1)^{m(f,0)} (a,f)$$
$$+ q^{(m-1)/2} \sum_{\substack{f \in F(m) \\ f(1)=0, f(m)\neq 2}} (-1)^{m(f,0)} (a,f)$$

and

$$\bar{A}_{\{s\}} T^\circ_{b(1)} T^\circ_{b(2)} \cdots T^\circ_{b(m)} (\bar{A}_{\{s\}} + \bar{A}_{\{t\}})$$
$$= \sum_{\substack{f \in F(m) \\ f(1)\neq 0}} (-1)^{m(f,0)} q^{|f|/2} (b,f)$$
$$= q^{(m+1)/2} \sum_{\substack{f \in F(m) \\ f(1)\neq 0, f(m)=2}} (-1)^{m(f,0)} (b,f)$$
$$+ q^{m/2} \sum_{\substack{f \in F(m) \\ f(1)\neq 0, f(m)\neq 2}} (-1)^{m(f,0)} (b,f).$$

Hence "$(\mathrm{A5})_{\{s\},\{s\}}$ and $(\mathrm{A5})_{\{s\},\{t\}}$" is equivalent to the following:

$$\text{(A6)} \qquad \sum_{\substack{f \in F(m) \\ f(1)=0, f(m)\neq 2}} (-1)^{m(f,0)} (a,f) = 0$$

$$\text{(A7)} \qquad \sum_{\substack{f \in F(m) \\ f(1)\neq 0, f(m)=2}} (-1)^{m(f,0)} (b,f) = 0$$

$$\text{(A8)} \qquad \sum_{\substack{f \in F(m) \\ f(1)=0, f(m)=2}} (-1)^{m(f,0)} (a,f) = \sum_{\substack{f \in F(m) \\ f(1)\neq 0, f(m)\neq 2}} (-1)^{m(f,0)} (b,f).$$

Since

$$\sum_{\substack{f \in F(m) \\ f(1)=0, f(m)\neq 2 \\ m(f,0)=k+1}} (-1)^{m(f,0)} (a,f)$$
$$= (-1)^{k+1} \binom{m-1-k}{k} (a(2), 1)(a(3), \dot{1}) \cdots (a(m-2k), 1),$$

(A6) is equivalent to

(A9)

$$\sum_{k=0}^{[(m-1)/2]} (-1)^k \binom{m-1-k}{k} (a(2), 1)(a(3), 1) \cdots (a(m-2k), 1) = 0.$$

Since

$$\sum_{\substack{f \in F(m) \\ f(1) \neq 0, f(m)=2 \\ m(f,0)=k}} (-1)^{m(f,0)} (b,f)$$
$$= (-1)^k \binom{m-1-k}{k} (b(1), 1)(b(2), 1) \cdots (b(m-2k-1), 1),$$

the equality (A7) is also equivalent to (A9). We have

$$\sum_{\substack{f \in F(m) \\ f(1)=0, f(m)=2 \\ m(f,0)=k+1}} (-1)^{m(f,0)} (a,f)$$
$$= \sum_{\substack{g \in F(m-2) \\ g(1) \neq 0, g(m-2) \neq 2 \\ m(g,0)=k}} (-1)^{k+1} (a(1), 0)(b, g)(a(m), 2).$$

By (A2), every summand of the right-hand side is of the form $\pm(b(1), 1)$ $(b(2), 1)$.... Hence the right-hand side is equal to

$$\sum_{\substack{g \in F(m-2) \\ g(1) \neq 0, g(m-2) \neq 2 \\ m(g,0)=k}} (-1)^{k+1} (a(0), 2)(a(1), 0)(b, g)$$
$$= \sum_{\substack{f \in F(m) \\ f(1)=2, f(m) \neq 2 \\ m(f,0)=k+1}} (-1)^{m(f,0)} (b,f).$$

Hence (A8) is equivalent to

$$\sum_{\substack{f \in F(m) \\ f(1)=1, f(m) \neq 2}} (-1)^{m(f,0)} (b,f) = 0.$$

Since

$$\sum_{\substack{f \in F(m) \\ f(1)=1, f(m) \neq 2 \\ m(f,0)=k}} (-1)^{m(f,0)} (b,f)$$
$$= (-1)^k \binom{m-1-k}{k} (b(1), 1)(b(2), 1) \cdots (b(m-2k), 1),$$

(A8) is also equivalent to

$$\sum_{k=0}^{[(m-1)/2]} (-1)^k \binom{m-1-k}{k} (b(1), 1)(b(2), 1) \cdots (b(m-2k), 1) = 0,$$

which is a consequence of (A9). Hence $(\mathrm{A}5)_{\{s\}}$ is equivalent to (A9), which can be rewritten as $(\mathrm{A}4)_b$.

Acknowledgment

The author would like to thank the referee for several useful suggestions. The content of Section 2.17 is due to the referee.

References

1. D. Alvis and G. Lusztig, The representations and generic degrees of the Hecke algebra of type H_4, *J. Reine Angew. Math.* **336** (1982), 201–212.
2. C. T. Benson and C. W. Curtis, On the degrees and rationality of certain characters of finite Chevalley groups, *Trans. Amer. Math. Soc.* **165** (1972), 251–273; **202** (1975), 405–406.
3. C. W. Curtis, N. Iwahori, and R. Kilmoyer, Hecke algebras and characters of parabolic type of finite groups with (B, N)-pairs, *I.H.E.S. Publ. Math.* **40** (1972), 81–116.
4. A. Gyoja, A generalized Poincaré series associated to a Hecke algebra of finite or p-adic Chevalley group, *Japan. J. Math.* **9** (1983), 87–111.
5. D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.
6. G. Lusztig, A class of irreducible representations of a Weyl group, *Indag. Math.* **41** (1979), 323–335.
7. G. Lusztig, On a theorem of Benson and Curtis, *J. Algebras* **71** (1981), 490–498.
8. G. Lusztig, A class of irreducible representations of a Weyl group, II, *Indag. Math.* **44** (1982), 219–226.
9. H. Naruse, Representations of Hecke algebras associated to W-graphs, *in* "Proceedings of Rokko Symposium" (in Japanese).